

A Note on Oscillating Kernels in Two Dimensions

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0. INTRODUCTION

Let $K_\theta(\xi) = (e^{i|\xi|}/|\xi|^{3/2}) \theta(\xi)$, where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$. In this paper, we determine by elementary methods, conditions on θ so that the Fourier transform of K_θ , \hat{K}_θ , does not belong to L^∞ . We assume throughout that θ , $\partial\theta/\partial\xi_1$, $\partial\theta/\partial\xi_2$, and $\partial^2\theta/\partial\xi_1\partial\xi_2$ are all continuous in \mathbb{R}^2 . These assumptions on θ can be considerably relaxed, but it would place too heavy a burden on the exposition.

Let $Tf = K_\theta * f$ with $\theta(\xi) = |\xi|^{-\varepsilon}$, $\varepsilon > 0$. In [1], the authors showed that the operators T map L^p onto L^p for $\frac{4}{3} < p < 4$; this was also done independently by the author of [2, 3]. This settles the (L^p, L^p) mapping problem for these kernels. While in [4, p. 12] it is shown for K_θ , θ radial and $\int_1^\infty (\theta(r)/r) dr = \infty$, that K_θ does not have a bounded Fourier transform and hence does not even map L^2 into L^2 . But it suggests for θ radial and $\int_1^\infty (\theta(r)/r) dr < \infty$ that K_θ does have a bounded Fourier transform. The solution to the mapping problem for the operators T_θ for general θ is the main interest of this paper.

Here we just consider the necessary part. That is, with very few restrictions on θ , we show that K_θ does not possess a bounded Fourier transform; in particular, we need not require that θ be radial. In Section 5, to give the reader an idea of how these methods work, we apply them to the cases $\theta \equiv 1$, $\theta \equiv \log^{-1}(2 + |\xi_1|)$, and $\theta \equiv \log^{-1}(2 + |\xi_1| + |\xi_2|)$.

1. PRELIMINARIES

We begin with (note throughout we take $t = (t_1, t_2)$)

PROPOSITION.

$$\begin{aligned} \int_c^d d\xi_2 \int_a^b d\xi_1 f(\xi) g(\xi) &= f(b, d) \int_c^d dt_2 \int_a^b dt_1 g(t) \\ &\quad - \int_a^b d\xi_1 \frac{\partial f}{\partial \xi_1}(\xi_1, d) \int_a^{\xi_1} dt_1 \int_c^d dt_2 g(t) \end{aligned}$$

$$\begin{aligned}
& - \int_c^d d\xi_2 \frac{\partial f}{\partial \xi_2}(b, \xi_2) \int_a^b dt_1 \int_c^{t_2} dt_2 g(t) \\
& + \int_c^d d\xi_2 \int_a^b d\xi_1 \frac{\partial^2 f}{\partial \xi_2 \partial \xi_1} \int_a^{t_1} dt_1 \int_c^{t_2} dt_2 g(t).
\end{aligned}$$

Remark 1. One should assume what is necessary so that all the expressions in the proposition make sense, for example, f , $\partial f/\partial \xi_1$, $\partial f/\partial \xi_2$, $\partial^2 f/\partial \xi_2 \partial \xi_1$, and g all exist and are continuous on $\mathbb{R} = [a, b] \times [c, d]$.

Proof of Proposition. We use two applications of integration by parts. We first note that

$$\begin{aligned}
\int_c^d d\xi_2 f g &= f(\xi_1, d) \int_c^d d\xi_2 g - \int_c^d d\xi_2 \frac{\partial f}{\partial \xi_2} \int_c^{t_2} dw g(\xi_1, w) \\
&= \text{I} + \text{II}.
\end{aligned}$$

For I, take

$$\begin{aligned}
r &= f(\xi_1, d), & ds &= d\xi_1 \int_c^d d\xi_2 g, \\
dr &= \frac{\partial f}{\partial \xi_1}(\xi_1, d) d\xi_1, & s &= \int_a^{t_1} dv \int_c^d d\xi_2 g(v, \xi_2);
\end{aligned}$$

For II, take

$$\begin{aligned}
r &= \partial f/\partial \xi_2, & ds &= -d\xi_1 \int_c^{t_2} dw g(\xi_1, w), \\
dr &= \frac{\partial^2 f}{\partial \xi_2 \partial \xi_1} d\xi_1, & s &= - \int_a^{t_1} dt_1 \int_c^{t_2} dt_2 g(t_1, t_2);
\end{aligned}$$

and now the proof of the proposition follows.

LEMMA 1 (Mean value result). *Let \mathbb{R} be the rectangle $[a, b] \times [c, d]$ and f a real-valued function. If f , $\partial f/\partial \xi_1$, $\partial f/\partial \xi_2$, and $\partial^2 f/\partial \xi_2 \partial \xi_1$ do not change sign in \mathbb{R} , then there are subrectangles $\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3, \mathbb{R}_4 \subseteq \mathbb{R}$ such that*

$$\begin{aligned}
\iint_{\mathbb{R}} f g &= f(P_1) \left[\iint_{\mathbb{R}_1} g - \iint_{\mathbb{R}_2} g - \iint_{\mathbb{R}_3} g + \iint_{\mathbb{R}_4} g \right] - f(P_2) \left[\iint_{\mathbb{R}_2} g + \iint_{\mathbb{R}_4} g \right] \\
&\quad - f(P_3) \left[\iint_{\mathbb{R}_3} g + \iint_{\mathbb{R}_4} g \right] + f(P_4) \iint_{\mathbb{R}_4} g,
\end{aligned}$$

where P_1, P_2, P_3 , and P_4 are the four vertices of \mathbb{R} .

Proof. By the second mean value theorem for integrals we get,

$$\int_a^b d\xi_1 \frac{\partial f}{\partial \xi_1}(\xi_1, d) \int_a^{\xi_1} dt_1 \int_c^d dt_2 g(t) = (f(b, d) - f(a, d)) \iint_{\mathbb{R}_2} g,$$

$$\int_c^d d\xi_2 \frac{\partial f}{\partial \xi_2}(b, \xi_2) \int_a^b dt_1 \int_c^{\xi_2} dt_2 g(t) = (f(b, d) - f(b, c)) \iint_{\mathbb{R}_3} g,$$

and

$$\int_c^d d\xi_2 \int_a^b d\xi_1 \frac{\partial^2 f}{\partial \xi_2 \partial \xi_1} \int_a^{\xi_1} dt_1 \int_a^{\xi_2} dt_2 g(t)$$

$$= (f(b, d) - f(b, c) - f(a, d) + f(a, c)) \iint_{\mathbb{R}_4} g,$$

and now by the Proposition the proof is complete.

LEMMA 2. Let $\mathbb{R} = [a, b] \times [c, d]$, $\phi(\xi_1, \xi_2) = p(\xi_1, \xi_2) - \xi \cdot x$, where p is a real-valued function, $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2$, $\frac{1}{2}|a| \leq |b| \leq 2|a|$, and $|c| + |d| \leq 2|a|$. Now suppose $(\partial\phi/\partial\xi_1)^{-1}$ satisfies the hypothesis of Lemma 1 (à la $f(\xi)$) on the rectangle \mathbb{R} . Now suppose $|\partial\phi/\partial\xi_1| \geq c_1 > 0$ for $(\xi_1, \xi_2) \in \mathbb{R}$, c_1 a constant. Then for $\alpha > 0$,

$$\left| \iint_{\mathbb{R}} \frac{e^{i\phi}}{|\xi|^{2-\alpha}} \right| \leq \frac{B}{c_1 |b|^{2-\alpha}} \left| \int_{c'}^{d'} e^{i\phi(A, \xi_2)} d\xi_2 \right|,$$

where $[c', d'] \subseteq [c, d]$ and B is a positive constant that depends only on α , but not on \mathbb{R} and not on x . Here A is some fixed number between a and b , but it may depend on x .

Proof.

$$I = \left| \iint_{\mathbb{R}} \frac{e^{i\phi}}{|\xi|^{2-\alpha}} \right| = \left| \int_c^d d\xi_1 \int_a^b d\xi_2 (\dots) \frac{(\partial\phi/\partial\xi_1)}{(\partial\phi/\partial\xi_1)} \right|.$$

Now note that $|\xi|^{\alpha-2}$ satisfies the hypothesis of Lemma 1 and hence,

$$I \leq B |b|^{\alpha-2} \left| \int_{c'}^{d'} d\xi_2 \int_{a'}^{b'} d\xi_1 \frac{e^{i\phi}(\partial\phi/\partial\xi_1)}{(\partial\phi/\partial\xi_1)} \right|,$$

where $[a', b'] \times [c', d'] \subseteq \mathbb{R}$. Now since $(\partial\phi/\partial\xi_1)^{-1}$ satisfies Lemma 1, we get (note $\partial\phi/\partial\xi_1$ stays one sign in \mathbb{R})

$$I \leq B \frac{|b|^{\alpha-2}}{c_1} \left| \int_{c''}^{d''} d\xi_2 \int_{a''}^{b''} d\xi_1 e^{i\phi} \partial\phi/\partial\xi_1 \right|$$

$$\leq \frac{B |b|^{\alpha-2}}{c_1} \left| \int_{c''}^{d''} d\xi_2 e^{i\phi(A, \xi_2)} \right|,$$

which completes the proof.

2. BASIC ESTIMATES

We begin this section with the key lemma of the paper.

LEMMA 3. Let $S(T, V) = |\int_0^V dt_1 \int_0^T dt_2 (e^{i(|t|-t_1)}/|t|^{3/2})|$, where $|t| = (t_1^2 + t_2^2)^{1/2}$. Set $U = \min(V, T)$; then

- (i) $S(T, V) \leq B\{U^{-1/2} + \log(2 + U)\}$ and
 (ii) $\int_0^V dt_1 \int_0^T dt_2 (\sin(|t| - t_1)/|t|^{3/2}) \geq B \log U$ for $U \geq 10$, B a positive constant independent of U .

Proof. We first note that

$$\left| \int_T^V dt_1 \int_0^T dt_2 \frac{e^{i(|t|-t_1)}}{|t|^{3/2}} \right| \leq B \quad \text{for } 0 \leq T \leq V, \quad (1)$$

and

$$\left| \int_0^V dt_1 \int_V^T dt_2 \frac{e^{i(|t|-t_1)}}{|t|^{3/2}} \right| \leq B/V^{1/2} \quad \text{for } 0 \leq V \leq T. \quad (2)$$

To see (1) we set $\phi = |t| - t_1$ and note

$$\partial^2 \phi / \partial t_2^2 = t_1^2 / |t|^3 \geq BT^2 / (T^2 + t_1^2)^{3/2} \geq BT^2 / (T^2 + T^2)^{3/2}.$$

And so by a one-dimensional version of Vander Corput we get

$$\int_T^V dt_1 \left| \int_0^T dt_2 (e^{i(|t|-t_1)}/|t|^{3/2}) \right| \leq BT^{1/2}/T^{1/2}.$$

To see (2) with ϕ as above, we note $\partial \phi / \partial t_2 = t_2 / |t| \geq V / (t^2 + V^2)^{1/2} \geq \frac{1}{2}\sqrt{2}$. Hence,

$$\int_0^V dt_1 \left| \int_V^T dt_2 \frac{e^{i\phi}}{|t|^{3/2}} \frac{\partial \phi / \partial t_2}{\partial \phi / \partial t_2} \right| \leq B \int_0^V \frac{dt_1}{(t_1^2 + V^2)^{3/4}} \leq \frac{BV}{V^{3/2}}.$$

Note that (2) is bounded for V bounded.

Now to continue the proof we note that

$$\begin{aligned} \int_0^U dt_1 \int_0^U dt_2 \frac{e^{i(|t|-t_1)}}{|t|^{3/2}} &= \int_0^{\pi/2} d\theta \int_0^U dr \frac{e^{ir(1-\cos\theta)}}{r^{1/2}} \\ &+ \int_0^U dt_1 \int_{\sqrt{U^2-t_1^2}}^U dt_2 \frac{e^{i(|t|-t_1)}}{|t|^{3/2}} = \text{I} + \text{II}. \end{aligned} \quad (3)$$

To estimate II we get

$$\partial\phi/\partial t_2 = \frac{t_2}{|t|} \geq \frac{(U^2 - t_1^2)^{1/2}}{(t_1^2 + (U^2 - t_1^2))^{1/2}} = \frac{(U^2 - t_1^2)^{1/2}}{U}.$$

And now

$$\begin{aligned} |\text{II}| &\leq \int_0^U dt_1 \left| \int_{\sqrt{U^2 - t_1^2}}^U dt_2 \frac{e^{i\phi}}{|t|^{3/2}} \frac{\partial\phi/\partial t_2}{\partial\phi/\partial t_2} \right| \\ &\leq \frac{BU}{U^{3/2}} \int_0^U \frac{dt_1}{(U^2 - t_1^2)^{1/2}} = BU^{-1/2}. \end{aligned} \quad (4)$$

Then to estimate I we have

$$\text{I} = \int_0^{\pi/2} \frac{d\theta}{(1 - \cos \theta)^{1/2}} \int_0^{U(1 - \cos \theta)} dr \frac{e^{ir}}{r^{1/2}}$$

and we note that

$$|\text{I}| \leq BU^{1/2}. \quad (5)$$

And also for $U \geq 2$ we get

$$\begin{aligned} \text{I} &= \int_0^{\sqrt{2/U}} \frac{d\theta}{(1 - \cos \theta)^{1/2}} \int_0^{U(1 - \cos \theta)} \frac{dr}{r^{1/2}} e^{ir} \\ &\quad + \int_{\sqrt{2/U}}^{\pi/2} \frac{d\theta}{(1 - \cos \theta)^{1/2}} \int_0^{U(1 - \cos \theta)} \frac{dr}{r^{1/2}} e^{ir}. \end{aligned}$$

Since $1 - \cos \theta \geq \frac{1}{2}\theta^2 - \frac{1}{24}\theta^4 \geq \frac{3}{8}\theta^2$ for $0 \leq \theta \leq \pi/2$ and $1 - \cos \theta \leq \frac{1}{2}\theta^2$, we get

$$|\text{I}| \leq B(1 + \log U). \quad (6)$$

Also note that

$$\begin{aligned} \int_0^T dr \frac{\sin r}{r^{1/2}} &\geq 0 && \text{for } T \geq 0, \\ &\geq \int_0^\pi dr \sin r \left\{ \frac{1}{r^{1/2}} - \frac{1}{(r + \pi)^{1/2}} \right\} > 0, && \text{for } T \geq \pi. \end{aligned}$$

Then for $U \geq 10$,

$$\begin{aligned} & \int_0^{\pi/2} \frac{d\theta}{(1 - \cos \theta)^{1/2}} \int_0^{U(1 - \cos \theta)} dr \frac{\sin r}{r^{1/2}} \\ & \geq \int_{(8\pi/3U)^{1/2}}^{\pi/2} \frac{d\theta}{(1 - \cos \theta)^{1/2}} \int_0^{U(1 - \cos \theta)} dr \frac{\sin r}{r^{1/2}} \\ & \geq B \int_{(8\pi/3U)^{1/2}}^{\pi/2} \frac{d\theta}{(1 - \cos \theta)^{1/2}} \geq B \int_{(8\pi/3U)^{1/2}}^{\pi/2} \frac{d\theta}{\theta} \\ & \geq B \log U. \end{aligned} \quad (7)$$

Now putting together our upper estimates for I and II, using (5) for small U and (6) for large U , we get part (i) of the lemma. Now since

$$\begin{aligned} & \left(\int_0^V dt_1 \int_0^T dt_2 - \int_0^U dt_1 \int_0^U dt_2 \right) \left(\frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right) \\ & = \int_U^V dt_1 \int_0^U dt_2 (\dots), \quad \text{for } V \geq T, \\ & = \int_0^U dt_1 \int_U^T dt_2 (\dots), \quad \text{for } T \geq V. \end{aligned} \quad (8)$$

And since $U \geq 10$ in (ii) of the lemma, then from (1), (2), and (8) the proof of (ii) follows from the estimate given in (7).

Remrk 2. We need the estimate in Lemma 3(i) generalized so it includes all the kernels $K_\theta(t)$, so we note that by the Proposition, with $f = \theta$ and $g = e^{i(|t| - t_1)}/|t|^{3/2}$,

$$\begin{aligned} & \left| \int_0^V dt_1 \int_0^T dt_2 K_\theta(t) e^{-it_1} \right| \\ & \leq B \left\{ |\theta(V, T)| \left| \int_0^V dt_1 \int_0^T dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right| \right. \\ & \quad + \int_0^V d\xi_1 \left| \frac{\partial \theta}{\partial \xi_1}(\xi_1, T) \right| \left| \int_0^{t_1} dt_1 \int_0^T dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right| \\ & \quad + \int_0^T d\xi_2 \left| \frac{\partial \theta}{\partial \xi_2}(V, \xi_2) \right| \left| \int_0^V dt_1 \int_0^{t_2} dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right| \\ & \quad \left. + \int_0^T d\xi_2 \int_0^V d\xi_1 \left| \frac{\partial^2 \theta}{\partial \xi_2 \partial \xi_1} \right| \left| \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right| \right\}. \end{aligned}$$

LEMMA 4. For $M \geq 2$,

$$\begin{aligned} & \left| \int_0^M d\xi_1 \int_0^M d\xi_2 K_\theta(\xi) e^{-i\xi_1(1-x_1) - \xi_2 x_2} \right| \\ & \leq B \left\{ M(|1-x_1| + |x_2|) \left| \int_0^M dt_1 \int_0^M dt_2 K_\theta(t) e^{-it_1} \right| \right. \\ & \quad + (|x_1-1| + |x_2|) \left(\int_0^M d\xi_1 \left| \int_0^{\xi_1} dt_1 \int_0^M dt_2 K_\theta(t) e^{-it_1} \right| \right. \\ & \quad \left. \left. + \int_0^M d\xi_2 \left| \int_0^{\xi_2} dt_1 \int_0^{\xi_2} dt_2 K_\theta(t) e^{-it_1} \right| \right) \right. \\ & \quad \left. + |x_2| |x_1-1| \int_0^M d\xi_2 \int_0^M d\xi_1 \left| \int_0^{\xi_1} dt_1 \int_0^{\xi_2} dt_2 K_\theta(t) e^{-it_1} \right| \right\}. \end{aligned}$$

Proof. We use the Proposition (in Section 1), where we set $f = 1 - e^{i(\xi_1(1-x_1) - \xi_2 x_2)}$ and $g(\xi) = K_\theta(\xi) e^{-i\xi_1}$ and the fact that $|1 - e^{i(M(1-x_1) - Mx_2)}| \leq M(|1-x_1| + |x_2|)$.

3. THE MAIN THEOREM

We begin this section by setting

$$G(M) = \int_0^M d\xi_1 \int_0^M d\xi_2 K_\theta(\xi) e^{-i\xi_1}$$

and

$$H(M) = \int_0^M d\xi_1 \int_0^M d\xi_2 K_\theta(\xi) e^{-i\xi \cdot x},$$

where $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2$. We note that when $\theta \equiv 1$ it follows by Lemma 3(ii) that $|G(M)| \geq B \log M$ for $M \geq 10$. From here on we shall consider only those functions θ for which $\lim_{M \rightarrow \infty} |G(M)| = \infty$. Actually $G(M)$ and $H(M)$ both depend on θ , but for simplicity we shall retain our notation. Also note that H depends on $x = (x_1, x_2)$ as well.

We first observe that $G(M) \approx H(M)$ for $x = (x_1, x_2)$ close to $(1, 0)$, so in order to show that $\hat{K}_\theta \notin L^\infty$, we shall need for those x 's to properly estimate $|H(N) - H(M)|$ for large N . Our theorem in this section will handle this problem. In Section 5, we apply this method to concrete examples.

We set (note θ is held fixed) $r(M) = \text{Min}\{M 2^{\log |G(M)|/2}, M^{5/4}\}$; $s(M) = M^{1/2} 2^{1/2(M^{1/2} \log |G(M)|)}$. Note that $\lim_{M \rightarrow \infty} M/s(M) = \lim_{M \rightarrow \infty} M/r(M) = 0$. Furthermore, we set

$$Z_1(M, N) = \theta(M, N) + \int_0^M d\xi_1 \left| \frac{\partial \theta}{\partial \xi_1}(\xi_1, N) \right| + \int_M^N d\xi_2 \left| \frac{\partial \theta}{\partial \xi_2}(M, \xi_2) \right| \\ + \int_M^N d\xi_2 \int_0^M d\xi_1 \left| \frac{\partial^2 \theta}{\partial \xi_2 \partial \xi_1} \right|$$

and

$$Z_2(M, N) = \theta(N, M) + \theta(N, N) + \int_0^N d\xi_2 \left| \frac{\partial \theta}{\partial \xi_2}(N, \xi_2) \right| \\ + \int_0^N d\xi_2 \int_M^N d\xi_1 \left| \frac{\partial^2 \theta}{\partial \xi_2 \partial \xi_1} \right| \\ + \int_M^N d\xi_1 \left(\left| \frac{\partial \theta}{\partial \xi_1}(\xi_1, M) \right| + \left| \frac{\partial \theta}{\partial \xi_1}(\xi_1, N) \right| \right).$$

THEOREM. Let $H(\cdot)$, $Z_1(\cdot, \cdot)$, $Z_2(\cdot, \cdot)$, $r(\cdot)$, and $s(\cdot)$ be as defined above. Note that $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2$. If $1/2r(M) \leq x_1 - 1 \leq 1/r(M)$ and $1/2s(M) \leq (-x_2) \leq 1/s(M)$, then

$$|H(N) - H(M)| \leq B \{ (Z_1(M, N)/M^{1/2}) + Z_2(M, N) \log |G(M)| \}$$

for $N \geq M \geq 10$. Here B is a positive constant independent of M , N , x_1 , and x_2 (for x_1, x_2 as in the theorem).

Proof. We note that

$$H(N) - H(M) = \int_0^M d\xi_1 \int_M^N d\xi_2 (\cdots) + \int_M^N d\xi_1 \int_0^M d\xi_2 (\cdots) \\ + \int_M^N d\xi_1 \int_M^N d\xi_2 (\cdots) \\ = \text{I} + \text{II} + \text{III}.$$

We shall apply the Proposition with $f = \theta$ and $g = (e^{i|\xi|}/|\xi|^{3/2}) e^{-i\xi \cdot x}$. For I we note (with g as above) that

$$\text{I} = \theta(M, N) \int_0^M d\xi_1 \int_M^N d\xi_2 g(\xi) - \int_0^M d\xi_1 \frac{\partial \theta}{\partial \xi_1}(\xi_1, N) \int_0^{\xi_1} dt_1 \int_M^N dt_2 g(t) \\ - \int_M^N d\xi_2 \frac{\partial \theta}{\partial \xi_2}(M, \xi_2) \int_0^M dt_1 \int_M^{\xi_2} dt_2 g(t) \\ + \int_M^N d\xi_2 \int_0^M d\xi_1 \frac{\partial^2 \theta}{\partial \xi_2 \partial \xi_1} \int_0^{\xi_1} dt_1 \int_M^{\xi_2} dt_2 g(t).$$

By Lemma 5 (which appears in Section 4) we get, $|\text{I}| \leq (B/M^{1/2}) Z_1(M, N)$.

In order to estimate II, we note again (with f and g as above) that

$$\begin{aligned} \text{II} = & \theta(N, M) \int_M^N d\xi_1 \int_0^M d\xi_2 g(\xi) \\ & - \int_M^N d\xi_1 \frac{\partial \theta}{\partial \xi_1}(\xi_1, M) \int_M^{\xi_1} dt_1 \int_0^M dt_2 g(t) \\ & - \int_0^M d\xi_2 \frac{\partial \theta}{\partial \xi_2}(N, \xi_2) \int_M^N dt_1 \int_0^{\xi_2} dt_2 g(t) \\ & + \int_0^M d\xi_2 \int_M^N d\xi_1 \frac{\partial^2 \theta}{\partial \xi_2 \partial \xi_1} \int_M^{\xi_1} dt_1 \int_0^{\xi_2} dt_2 g(t). \end{aligned}$$

It follows by Lemma 7 (which appears in Section 4) that

$$|\text{II}| \leq B(\log |G(M)| + 1) Z_2(M, N)$$

as long as $1/2r(M) \leq x_1 - 1 \leq 1/r(M)$ and $1/2s(M) \leq (-x_2) \leq 1/s(M)$.

In order to estimate III, we again note that

$$\begin{aligned} \text{III} = & \theta(N, N) \int_M^N d\xi_1 \int_M^N d\xi_2 g(\xi) \\ & - \int_M^N d\xi_1 \frac{\partial \theta}{\partial \xi_1}(\xi_1, N) \int_M^{\xi_1} dt_1 \int_M^N dt_2 g(t) \\ & - \int_M^N d\xi_2 \frac{\partial \theta}{\partial \xi_2}(N, \xi_2) \int_M^N dt_1 \int_M^{\xi_2} dt_2 g(t) \\ & + \int_M^N d\xi_2 \int_M^N d\xi_1 \frac{\partial^2 \theta}{\partial \xi_2 \partial \xi_1} \int_M^{\xi_1} dt_1 \int_M^{\xi_2} dt_2 g(t). \end{aligned}$$

Now by Lemma 6 (which appears in Section 4) we get

$$|\text{III}| \leq B(1 + \log |G(M)|) Z_2(M, N)$$

for $x = (x_1, x_2)$ as in the hypothesis. The proof of the theorem is now complete.

4. PROOFS OF LEMMAS 5-7

Throughout this section we set $\phi = |\xi| - \xi \cdot x$, and throughout the paper we assume with $x = (x_1, x_2)$ that $x_1 \geq 1$ and $x_2 \leq 0$. We begin with

LEMMA 5. *We get*

$$\left| \int_M^T d\xi_2 \int_0^V d\xi_1 \frac{e^{i\phi}}{|\xi|^{3/2}} \right| \leq BM^{-1/2}, \quad V \leq M \leq T,$$

B a positive constant independent of M , T , V , and x .

Proof.

$$\partial\phi/\partial\xi_2 = (\xi_2/|\xi|) - x_2 \geq M/(\xi_1^2 + M^2)^{1/2} \geq 1/\sqrt{2}$$

and thus

$$\left| \int_0^V d\xi_1 \int_M^T d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_2)}{(\partial\phi/\partial\xi_2)} \right| \leq \frac{B}{M^{3/2}} M.$$

LEMMA 6. *For M sufficiently large,*

$$\begin{aligned} & \left| \int_M^N d\xi_1 \int_M^T d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \right| \\ & \leq B \left(1 + \log |G(M)| + \frac{1}{(x_1 - 1)r(M)} + \frac{1}{|x_2|s(M)} \right), \end{aligned}$$

where here $N \geq M$, $T \geq M$. Also, $r(M)$ and $s(M)$ are as defined in Section 3.

Proof.

$$\begin{aligned} \int_M^N d\xi_1 \int_M^T d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} &= \sum_{k=0}^{\infty} \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_M^{M2^{k/2}} d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \\ &+ \sum_{k=0}^{\infty} \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{M2^{k/2}}^T d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \\ &= \text{I} + \text{II}, \end{aligned}$$

where \sum' indicates that $M2^k$ is a proper endpoint as long as $M2^k \leq N$, otherwise we use N ; and similarly for $M2^{k/2}$ and T .

To estimate term I we note that

$$\left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_M^{M2^{k/2}} d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_2)}{(\partial\phi/\partial\xi_2)} \right| \leq \frac{B2^k}{(M2^k)^{1/2}} = \frac{B2^{k/2}}{M^{1/2}}$$

since $\partial\phi/\partial\xi_2 = (\xi_2/|\xi|) - x_2 \geq M/(M^2 2^{2k+2} + M^2)^{1/2} \geq B/2^k$.

Furthermore, since $\partial\phi/\partial\xi_1 = (\xi_1/|\xi|) - x_1 = (\xi_1/|\xi|) - 1 + 1 - x_1$ with $x_1 \geq 1$, we get $|\partial\phi/\partial\xi_1| \geq x_1 - 1$, and that implies

$$\left| \int_M^{M2^{k/2}} d\xi_2 \int_{M2^k}^{M2^{k+1}} d\xi_1 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_1)}{(\partial\phi/\partial\xi_1)} \right| \leq \frac{BM2^{k/2}}{(x_1 - 1)(M2^k)^{3/2}} \\ = \frac{B}{(x_1 - 1)M^{1/2}2^k}.$$

Now choose t so that $(2^t/M)^{1/2} = \log |G(M)|$ (note M is sufficiently large and so $\log |G(M)| > 1$), hence

$$|I| \leq \frac{B}{M^{1/2}} \sum_{k=0}^t 2^{k/2} + \frac{B}{(x_1 - 1)M^{1/2}} \sum_{k=t}^{\infty} \frac{1}{2^k} \\ \leq B \left(\log |G(M)| + \frac{1}{(x_1 - 1)M^{3/2} \log^2 |G(M)|} \right).$$

To estimate term II we note that

$$\left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{M2^{k/2}}^T d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_2)}{(\partial\phi/\partial\xi_2)} \right| \leq \frac{B2^{k/2}}{(M2^k)^{1/2}} = \frac{B}{M^{1/2}}$$

since $\partial\phi/\partial\xi_2 = \xi_2/|\xi| - x_2 \geq M2^{k/2}/(M^22^{2k+2} + M^22^k)^{1/2} \geq B/2^{k/2}$. Also

$$\left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{M2^{k/2}}^T d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_2)}{(\partial\phi/\partial\xi_2)} \right| \leq \frac{B}{|x_2| (M2^k)^{1/2}},$$

hence

$$|II| \leq B \left(\sum_{k=0}^t \frac{1}{M^{1/2}} + \sum_{k=t}^{\infty} \frac{1}{|x_2| M^{1/2} 2^{k/2}} \right) \leq B \left(\frac{t}{M^{1/2}} + \frac{1}{|x_2| (M2^t)^{1/2}} \right).$$

Now select t so that $t/M^{1/2} = \log |G(M)|$. Then

$$|II| \leq B \left(\log |G(M)| + \frac{1}{|x_2| M^{1/2} 2^{(1/2)(M^{1/2} \log |G(M)|)}} \right).$$

Now putting the estimates for I and II together, we get the proof of the lemma.

The next lemma requires more care than the previous two, as we shall see.

LEMMA 7. For M sufficiently large and $0 \leq x_1 - 1 \leq 1/(8\sqrt{2})M$ we get

$$\left| \int_M^V d\xi_1 \int_0^T d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \right| \leq B \left(1 + \log |G(M)| + \frac{1}{(x_1 - 1)r(M)} \right)$$

for $T \leq M \leq V$ and $r(M)$ as defined in Section 3.

Proof. Let us note that

$$\int_V^\infty d\xi_1 \int_0^{V^{1/2}} d\xi_2 \frac{1}{|\xi|^{3/2}} \leq B. \quad (9)$$

We shall apply (9) when $V = M, M^2$. Because of (9) it suffices to consider only

$$\int_M^{M^2} d\xi_1 \int_{M^{1/2}}^M d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}}. \quad (10)$$

We note for $2^s = M$,

$$\begin{aligned} \int_M^{M^2} d\xi_1 \int_{M^{1/2}}^M d\xi_2(\dots) &= \sum_{k=0}^s \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{M^{1/2}}^{(M2^k)^{1/2}} d\xi_2(\dots) \\ &\quad + \sum_{k=0}^s \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{(M2^k)^{1/2}}^M d\xi_2(\dots) \\ &= \text{I} + \text{II}. \end{aligned} \quad (10')$$

Now we note from (9) ($V = M2^k$) that

$$\int_{M2^k}^{M2^{k+1}} d\xi_1 \left| \int_{M^{1/2}}^{(M2^k)^{1/2}} d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \right| \leq B,$$

and also that

$$\int_{M^{1/2}}^{(M2^k)^{1/2}} d\xi_2 \left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_1)}{(\partial\phi/\partial\xi_1)} \right| \leq \frac{B}{(x_1 - 1)} \frac{(M2^k)^{1/2}}{(M2^k)^{3/2}}.$$

Hence for term I we get

$$\begin{aligned} |\text{I}| &\leq \sum_{k=0}^{\log|G(M)|} |\dots| + \sum_{k=\log|G(M)|}^{\infty} |\dots| \\ &\leq B \log|G(M)| + \frac{B}{(x_1 - 1) M2^{\log|G(M)|}}. \end{aligned}$$

To estimate term II, we first note

$$\left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{(M2^k)^{1/2}}^M d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_2)}{(\partial\phi/\partial\xi_2)} \right| \leq \frac{B(M2^k)^{1/2}}{(M2^k)^{1/2}} = B \quad (11)$$

since $\partial\phi/\partial\xi_2 \geq \xi_2/|\xi| \geq (M2^k)^{1/2}/(M^2 2^{2k+2} + M2^k)^{1/2} \geq B/(M2^k)^{1/2}$. Now

$$\text{II} = \sum_{k=0}^{(1/2)s} (\dots) + \sum_{k=(1/2)s}^s (\dots) = \text{III} + \text{IV}.$$

To estimate IV we note that

$$\int_{(M2^k)^{1/2}}^M d\xi_2 \left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_1)}{(\partial\phi/\partial\xi_1)} \right| \leq \frac{B}{(x_1-1)} \frac{M}{(M2^k)^{3/2}}$$

and so

$$|\text{IV}| \leq \frac{B}{(x_1-1)M^{1/2}} \sum_{k=(1/2)s}^s (\dots) \leq \frac{B}{(x_1-1)M^{5/4}}.$$

Before we estimate III we note that

$$\int_{(M2^k)^{1/2}}^{M^{1/2}2^k} d\xi_2 \left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \frac{e^{i\phi}}{|\xi|^{3/2}} \frac{(\partial\phi/\partial\xi_1)}{(\partial\phi/\partial\xi_1)} \right| \leq \frac{B}{(x_1-1)M2^{k/2}}. \quad (12)$$

Now suppose $(\partial\phi/\partial\xi_1)^{-1}$ satisfies the hypothesis of Lemma 2. Then by Lemma 2 it follows ($k \leq (\frac{1}{2}s$)) that if A is some number between $M2^k$ and $M2^{k+1}$, then

$$\begin{aligned} & \left| \int_{M^{1/2}2^k}^M d\xi_2 \int_{M2^k}^{M2^{k+1}} d\xi_1 \frac{e^{i\phi}}{|\xi|^{3/2}} \right| \\ & \leq \frac{B}{(x_1-1)(M2^k)^{3/2}} \left| \int_{(M^{1/2}2^k)}^{(M)} d\xi_2 e^{i\phi(A, \xi_2)} \right| \\ & \leq \frac{B}{(x_1-1)(M2^k)^{3/2}} \left| \int d\xi_2 e^{i\phi(A, \xi_2)} \frac{(\partial\phi/\partial\xi_2)}{(\partial\phi/\partial\xi_2)} \right| \\ & \leq \frac{BM^{1/2}}{(x_1-1)(M2^k)^{3/2}} \end{aligned} \quad (13)$$

since $\partial\phi(A, \xi_2)/\partial\xi_2 \geq \xi_2/(A^2 + \xi_2^2)^{1/2} \geq BM^{1/2}2^k/M2^k = B/M^{1/2}$. Now by (11) we get

$$\begin{aligned} |\text{III}| & \leq \sum_{k=0}^{\log|G(M)|} |\dots| + \sum_{k=\log|G(M)|}^{(1/2)s} |\dots| \\ & \leq B \log|G(M)| + \sum_{k=\log|G(M)|}^{(1/2)s} |\dots|, \end{aligned}$$

while

$$\begin{aligned}
 & \sum_{k=\log|G(M)|}^{(1/2)s} \left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{(M2^k)^{1/2}}^M d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \right| \\
 & \leq \sum_{k=\log|G(M)|}^{(1/2)s} \left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{(M2^k)^{1/2}}^{M^{1/2}2^k} d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \right| \\
 & \quad + \sum_{k=\log|G(M)|}^{(1/2)s} \left| \int_{M2^k}^{M2^{k+1}} d\xi_1 \int_{M^{1/2}2^k}^M d\xi_2 \frac{e^{i\phi}}{|\xi|^{3/2}} \right| \\
 & \leq B \sum_{k=\log|G(M)|}^{(1/2)s} \left(\frac{1}{M(x_1-1)2^{k/2}} + \frac{1}{M(x_1-1)2^{k(3/2)}} \right) \\
 & \leq \frac{B}{M(x_1-1)2^{(1/2)\log|G(M)|}}.
 \end{aligned}$$

The last two steps result from (12) and (13). Putting together our estimates for III and IV, we complete the estimate for II. Then putting our estimates for I and II together, we shall have completed the proof once we show that $(\partial\phi/\partial\xi_1)^{-1}$ satisfies Lemma 2 (we needed that for (13)).

First set $f = -(\partial\phi/\partial\xi_1)^{-1}$; to show that f satisfies Lemma 2 it suffices to show that $\partial^2 f / \partial\xi_2 \partial\xi_1$ stays one sign in the rectangle $M2^k \leq \xi_1 \leq M2^{k+1}$ and $M^{1/2}2^k \leq \xi_2 \leq M$. But

$$\frac{\partial^2 f}{\partial\xi_2 \partial\xi_1} = - \left(\frac{\partial\phi}{\partial\xi_1} \right)^{-3} |\xi|^{-3} \xi_2 (-2\xi_1 \xi_2^2 - (\partial\phi/\partial\xi_1) |\xi| (2\xi_1^2 - \xi_2^2)). \quad (14)$$

Note $-\partial\phi/\partial\xi_1 = (x_1 - 1) + (|\xi| - \xi_1)/|\xi| \leq (x_1 - 1) + \frac{1}{2}\xi_2^2/\xi_1 |\xi|$. And since $2\xi_1^2 - \xi_2^2 \geq 0$ here, we get

$$\begin{aligned}
 & -2\xi_1 \xi_2^2 - (\partial\phi/\partial\xi_1) |\xi| (2\xi_1^2 - \xi_2^2) \\
 & \leq (x_1 - 1) |\xi| (2\xi_1^2 - \xi_2^2) - \xi_1 \xi_2^2 - \frac{1}{2}\xi_2^4/\xi_1.
 \end{aligned}$$

Once we show $(x_1 - 1) |\xi| 2\xi_1^2 \leq \xi_1 \xi_2^2$ we shall be finished. Note that $\xi_2^2/2 |\xi| \xi_1 \geq M2^k/2 \sqrt{2} (M2^{k+1})^2 = 1/M8 \sqrt{2}$, but by hypothesis $x_1 - 1 \leq 1/(8\sqrt{2}) M \leq \xi_2^2/2\xi_1 |\xi|$. Thus f satisfies Lemma 2 and the proof is complete.

5. APPLICATIONS

In this section we show that for $x \in \mathbb{R}^2$

$$\hat{K}_\theta(x) = \iint_{\mathbb{R}^2} K_\theta(\xi) e^{-i\xi \cdot x} d\xi$$

is not in L^∞ for various values of θ . We shall show that \hat{K}_θ is large for points $x = (x_1, x_2)$ where x_1 is near 1 and x_2 is near 0. And so it suffices to show that

$$\int_0^\infty d\xi_1 \int_0^\infty d\xi_2 K_\theta(\xi) e^{-i\xi \cdot x}$$

is arbitrarily large for these values of x . Furthermore, because of (1), (2), and Remark 2 (following Lemma 3) it suffices to show that $H(N)$ (as defined in Section 3) is arbitrarily large for these x 's.

As we pointed out in the Introduction, the boundedness of the Fourier transform of K_θ is already understood when θ is radial. In this paper, we wanted to show, by elementary methods, that for a "large class" of θ 's (θ not radial) $\hat{K}_\theta \notin L^\infty$. In order to give the reader an idea how these methods work, we shall apply them to the cases where $\theta \equiv 1$, $\theta \equiv \log^{-1}(2 + |\xi_1|)$, or $\theta \equiv \log^{-1}(2 + |\xi_1| + |\xi_2|)$.

Note that $G(M)$, $H(M)$, $s(M)$, and $r(M)$ were defined in Section 3. As stated there, we are only interested in those θ 's for which $|G(M)| \rightarrow \infty$ as $M \rightarrow +\infty$. Then by Lemma 4 with $|x - 1| \leq 1/r(M)$ and $|x_2| \leq 1/s(M)$ we get that

$$|G(M) - H(M)| \leq BM((1/s(M)) + (1/r(M))) |G(M)|, \quad (15)$$

keeping Remark 2 in mind. (We shall give a more complete discussion in the Corollary.) Hence $|H(M)| \geq B |G(M)|$ for these x 's. Now by the Theorem in Section 3 for $N \geq M \geq 10$, $1/2r(M) \leq x_1 - 1 \leq 1/r(M)$, and $1/2s(M) \leq (-x_2) \leq 1/s(M)$, we get

$$|H(M) - H(N)| \leq B\{(Z_1(M, N)/M^{1/2}) + Z_2(M, N) \log |G(M)|\}. \quad (16)$$

COROLLARY. Let $K_\theta(\xi) = (e^{i|\xi|}/|\xi|^{3/2}) \theta(\xi)$, where θ is one of the functions 1 , $\log^{-1}(2 + |\xi_1|)$, $\log^{-1}(2 + |\xi_1| + |\xi_2|)$. Then $\hat{K}_\theta \notin L^\infty$ or, equivalently, $K_\theta \notin L_2^2$. That implies $\hat{K}_\theta \notin M(H^p)$ for $p > 0$.

Proof. Let us begin with the case $\theta \equiv 1$. Here, $G(M) = \int_0^M d\xi_1 \int_0^M d\xi_2 (e^{i(|\xi| - \xi_1)}/|\xi|^{3/2})$ and by Lemma 3(i, ii) we get that $B_1 \leq |G(M)|/\log M \leq B_2$ for M large. So by Lemma 4 and (16) it follows that $|\hat{K}_\theta(x)| \geq B \log M$ as long as $1/2r(M) \leq x_1 - 1 \leq 1/r(M)$ and $1/2s(M) \leq (-x_2) \leq 1/s(M)$. For this case $\hat{K}_\theta \notin L^\infty$.

For $\theta = \log^{-1}(2 + |\xi_1|)$, by the Proposition (Section 1) with $f = \log^{-1}(2 + |\xi_1|)$ and $g = e^{i(|\xi| - \xi_1)}/|\xi|^{3/2}$, we get

$$\begin{aligned} G(M) = & \log^{-1}(2 + M) \int_0^M d\xi_1 \int_0^M d\xi_2 \frac{e^{i(|\xi| - \xi_1)}}{|\xi|^{3/2}} \\ & + \int_0^M d\xi_1 \frac{1}{(2 + \xi_1) \log^2(2 + \xi_1)} \int_0^{\xi_1} dt_1 \int_0^M dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \end{aligned} \quad (17)$$

Now by Lemma 3(ii) we get (for M large)

$$\begin{aligned} & \int_0^M d\xi_1 \frac{1}{(2 + \xi_1) \log^2(2 + \xi_1)} \int_0^{\xi_1} dt_1 \int_0^M dt_2 \frac{\sin(|t| - t_1)}{|t|^{3/2}} \\ & \geq B \int_0^M d\xi_1 \frac{\log \xi_1}{(2 + \xi_1) \log^2(2 + \xi_1)} \geq B \log(\log M). \end{aligned} \quad (18)$$

By applying Lemma 3(i) to (17) and by (18) we get $B_1 \leq |G(M)|/\log(\log M) \leq B_2$ for M large.

Next set $U = \min(V, T)$ and suppose U is large. Then

$$\begin{aligned} I &= \int_0^V d\xi_1 \frac{1}{(2 + \xi_1) \log^2(2 + \xi_1)} \left| \int_0^{\xi_1} dt_1 \int_0^T dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right| \\ &= \left(\int_0^U + \int_U^V \right) d\xi_1 \frac{1}{(2 + \xi_1) \log^2(2 + \xi_1)} \left| \int_0^{\xi_1} dt_1 \int_0^T dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right|. \end{aligned}$$

Now by Lemma 3(i) we get

$$\begin{aligned} I &\leq B \left\{ \int_0^U d\xi_1 \frac{(\xi_1^{-1/2} + \log \xi_1)}{(2 + \xi_1) \log^2(2 + \xi_1)} + \int_U^V d\xi_1 \frac{(U^{-1/2} + \log U)}{(2 + \xi_1) \log^2(2 + \xi_1)} \right\} \\ &\leq B \log(\log U). \end{aligned}$$

Furthermore, for $U \leq M$ and M large by Remark 2 we get that

$$\begin{aligned} & \left| \int_0^V dt_1 \int_0^T dt_2 K_\theta(t) e^{-it_1} \right| \\ & \leq B \left\{ 1 + \int_0^V d\xi_1 \frac{1}{(2 + \xi_1) \log^2(2 + \xi_1)} \left| \int_0^{\xi_1} dt_1 \int_0^T dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right| \right\} \\ & \leq B \log(\log M) = B |G(M)|. \end{aligned}$$

We now see why (15) holds. Our result follows from (16) once we observe that $Z_1(M, N) + Z_2(M, N) \leq B$.

Now we shall do the case with $\theta = \log^{-1}(2 + |\xi_1| + |\xi_2|)$. By the Proposition, with $f = \log^{-1}(2 + |\xi_1| + |\xi_2|)$ and $g = e^{i(|\xi_1| - \xi_1)}/|\xi_1|^{3/2}$, we get

$$\begin{aligned} G(M) &- \int_0^M d\xi_2 \int_0^M d\xi_1 \frac{1}{(2 + \xi_1 + \xi_2)^2 \log^2(2 + \xi_1 + \xi_2)} \\ & \times \int_0^{\xi_1} dt_1 \int_0^{\xi_2} dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \end{aligned}$$

$$\begin{aligned}
&= \log^{-1}(2 + 2M) \int_0^M d\xi_1 \int_0^M d\xi_2 \frac{e^{i(|\xi| - \xi_1)}}{|\xi|^{3/2}} \\
&\quad + \int_0^M \frac{d\xi_1}{(2 + \xi_1 + M) \log^2(2 + \xi_1 + M)} \int_0^{\xi_1} dt_1 \int_0^M dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \\
&\quad + \int_0^M \frac{d\xi_2}{(2 + \xi_2 + M) \log^2(2 + \xi_2 + M)} \int_0^M dt_1 \int_0^{\xi_2} dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \\
&\quad + \int_0^M d\xi_1 \int_0^M d\xi_2 \frac{2}{(2 + \xi_1 + \xi_2)^2 \log^3(2 + \xi_1 + \xi_2)} \\
&\quad \times \int_0^{\xi_1} dt_1 \int_0^{\xi_2} dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| G(M) - \int_0^M d\xi_2 \int_0^M d\xi_1 \frac{1}{(2 + \xi_1 + \xi_2)^2 \log^2(2 + \xi_1 + \xi_2)} \right. \\
&\quad \left. \times \int_0^{\xi_1} dt_1 \int_0^{\xi_2} dt_2 \frac{e^{i(|t| - t_1)}}{|t|^{3/2}} \right| \leq B.
\end{aligned} \tag{19}$$

But by Lemma 3(ii) we get

$$\begin{aligned}
&\int_0^M d\xi_1 \int_0^M d\xi_2 \frac{1}{(2 + \xi_1 + \xi_2)^2 \log^2(2 + \xi_1 + \xi_2)} \int_0^{\xi_1} dt_1 \int_0^{\xi_2} dt_2 \frac{\sin(|t| - t_1)}{|t|^{3/2}} \\
&\geq B \int_0^M d\xi_1 \int_0^M d\xi_2 \frac{\log(2 + \xi_1)}{(2 + \xi_1 + \xi_2)^2 \log^2(2 + \xi_1 + \xi_2)} \geq B \log(\log M),
\end{aligned}$$

B a positive constant independent of M . From (19) and Lemma 3(i) we get $B_1 \leq |G(M)|/\log(\log M) \leq B_2$. As before, by (15), (16), and the fact that $Z_1(M, N) + Z_2(M, N) \leq B$, we get our result. Note that in order to see (15) here we use Remark 2 and Lemma 3(i) just as we did for the previous example.

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